Section 4.6 Variation of Parameters

This a method to find y_p. It can be used for any problem, but at times the integration can be difficult. It will work when undetermined coefficients fail.

① Solve y_c. You will need information from y_c to find y_p.

original $= \underline{f(x)}$

$$y_c = c_1 y_1 + c_2 y_2 + \cdots$$

② use wronskians to find y_p.

$$y_p = u_1 y_1 + u_2 y_2 + \cdots$$

where u_n is a function of x (when $y = f(x)$)

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad \cdots$$

with $W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & \\ y_1'' & y_2'' & & \\ \vdots & \vdots & & \end{vmatrix}$ $\quad W_1 = \begin{vmatrix} 0 & y_2 & \cdots & y_n \\ \vdots & y_2' & & \vdots \\ \vdots & \vdots & & \vdots \\ f(x) & y_2^{(n-1)} & & y_n^{(n-1)} \end{vmatrix}$

Then integrate u_n' to find u_n. The final answer will be in the form

$$y_p = u_1 y_1 + u_2 y_2 + \cdots + u_n y_n$$

Example 2

Find a particular soln and general solution for
$y'' + y' + y = \sin^2 x$

$\boxed{y_c}$ $m^2 + m + 1 = 0$ $m = \frac{-1 \pm \sqrt{1 - 4(1)(1)}}{2(1)}$

$m = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$

so $y_c = C_1 e^{-\frac{1}{2}x} \cos \frac{\sqrt{3}}{2} x + C_2$

$\boxed{\text{Example 1}}$ $\boxed{\text{find } y_p \text{ using variation of parameters}}$

$$\boxed{y'' + 3y' + 2y = 4e^{x}}$$

$\boxed{y_c}$

$$m^2 + 3m + 2 = 0$$
$$(m+2)(m+1) = 0$$
$$m = -2 \qquad m = -1$$

$$y_c = c_1 \underbrace{e^{-2x}}_{y_1} + c_2 \underbrace{e^{-x}}_{y_2}$$

$\boxed{y_p}$

$$y_p = u_1 y_1 + u_2 y_2$$

$$W = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}$$

$$= (e^{-2x})(-e^{-x}) - (-2e^{-2x})(e^{-x})$$
$$= -e^{-3x} + 2e^{-3x} = \boxed{e^{-3x}}$$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ 4e^{x} & -e^{-x} \end{vmatrix} = -(e^{-x})(4e^{x}) = \boxed{-4}$$

$$W_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & 4e^{x} \end{vmatrix} = (e^{-2x})(4e^{x}) = \boxed{4e^{-x}}$$

$$u_1' = \frac{W_1}{W} = \frac{-4}{e^{-3x}}$$

$$u_1 = \int -4e^{3x}\,dx = \boxed{\frac{-4}{3}e^{3x}}$$

$\boxed{\text{example 1. (4.6) continued}}$

$$u_2' = \frac{W_2}{W} = \frac{4e^{-x}}{e^{-3x}} = 4e^{2x}$$

$$u_2 = \int 4e^{2x}\,dx = \boxed{2e^{2x}}$$

$$\text{so } y_p = u_1y_1 + u_2y_2$$
$$= \left(-\tfrac{4}{3}e^{3x}\right)\left(e^{-2x}\right) + \left(2e^{2x}\right)\left(e^{-x}\right)$$
$$= -\tfrac{4}{3}e^{x} + 2e^{x}$$
$$= -\tfrac{4}{3}e^{x} + \tfrac{6}{3}e^{x} = \boxed{\tfrac{2}{3}e^{x} = y_p}$$

$$y = y_c + y_p$$

$$\boxed{y = c_1e^{-2x} + c_2e^{-x} + \tfrac{2}{3}e^{x}}$$

Note: If initial conditions were specified, you would use that information at this point (after finding y_p).

Section 4.6 Variation of Parameters
Extra Practice Problems

Find the general solution of the second order DE using variation of parameters

① $y'' + 9y = \tan(3t)$

② $y'' - y = t + 3$

③ $y'' - 2y' + y = e^{t}$

④ $x'' + x = \tan^2 t$

⑤ $x'' + x = \tan^3 t$

⑥ $y'' + y = \tan t + \sin t + 1$

Answers

① $y = c_1 \cos 3t + c_2 \sin 3t - \frac{1}{9}\cos 3t \ln|\sec 3t + \tan 3t|$

② $y = c_1 e^{t} + c_2 e^{-t} - t - 3$

③ $y = c_1 e^{t} + c_2 te^{t} + \frac{1}{2}t^2 e^{t}$

④ $x = c_1 \cos t + c_2 \sin t - 2 + \sin t \ln|\sec t + \tan t|$

⑤ $x = c_1 \cos t + c_2 \sin t + \frac{[1 + \cos^2 t]}{3}$

⑥ $y = c_1 \cos t + c_2 \sin t + 1 - \frac{1}{2}t\cos t \ln|\sec t + \tan t| + \frac{1}{4}\sin t$